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Stochastic escape processes from a non-symmetric potential normal form

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Abstract. The lifetime from the normal form $\dot{X} = aX^2 + b + \sqrt{\epsilon}\xi(t)$ is analytically studied in terms of the escape time of leaving the unstable point $X = 0$. A perturbation theory, in the small noise parameter $\sqrt{\epsilon}$, is introduced to analyse the escape of the stochastic paths. We show that the first passage time density satisfies a scale transformation. The anomalous fluctuation of the phase-space variable $X(t)$ (when there is saturation in the potential of the normal form) is analytically calculated using an instanton-like approximation. An emphasis is placed on thermal explosions in order to exemplify a system undergoing hysteresis in a first-order non-equilibrium phase transition. We carried out Monte Carlo simulations showing excellent agreement with our theoretical predictions.

1. Introduction

The influence of noise on pattern-forming instabilities in non-equilibrium systems near the bifurcation point has received renewed attention, both experimentally and theoretically. Near the threshold the system can often be reduced to the study of the stochastic Landau equation (for the order parameter X), and the mesoscopic description (fluctuations) is solved using that stochastic differential equation (SDE) [1].

Nonlinear systems far from equilibrium exhibit a variety of instabilities when the appropriate control parameters are changed [1, 2]. By such changes of the control parameters the system can be placed in an unstable state. The system, in general, will relax to a metastable (or global) stationary state. This transient process is triggered by the noise ($O(\sqrt{\epsilon})$), while the statistical description of such a transient constitutes one of the main subjects of non-equilibrium statistical mechanics. A detailed description of the relaxation process depends on the nature of the instability involved. Unstable states appear in first-order-like instabilities at the end point of hysteresis cycles. Typical cases are those possessing the symmetry transformation $X \longrightarrow -X$ in the relaxation from $X = 0$. A theory for the relaxation at a subcritical pitchfork bifurcation when there is such an inversion symmetry has recently been successfully introduced [3]. This approach is based on the fact that each stochastic path (up to $O(\sqrt{\epsilon})$) can be approximated systematically with a suitable perturbation on the deterministic path. Therefore, the lifetime from an unstable state can be studied in terms of the random escape times which, in fact, are governed by those approximated stochastic paths. This theory allows us, in principle, to find the lifetime of any unstable state (i.e. the passage time to some macroscopic value $X \approx O(1)$). The lack of an initial Gaussian regime does not pose any restrictions for determining the statistical properties of the lifetime from an unstable state [3, 4]. Here we will focus on the special case (near the critical point of the system) where inversion symmetry in the potential of the SDE is lost.

The physical motivation to study the stochastic Landau equation,

$$\dot{X} = aX^2 + b + \sqrt{\epsilon}\xi(t) \tag{1}$$

lies in the fact that at the critical point for thermal explosive systems [5], the stochastic Semenov model takes the normal form given in equation (1), where $\xi(t)$ is a zero-mean Gaussian white noise and X represents the order parameter of the system (the temperature). In general, the approach presented here can also be used to study unstable states other than $X = 0$. Here we will only be interested in the potential $U(X) = -aX^3/3 - bX$, for a and b positive constants, as obtained from equation (1).

It has been shown that the mean first passage time (MFPT) of equation (1) gives information about the thermal explosion times of a homogeneous physicochemical reactor [6, 7]. In a previous paper [5] we were able to show that the switching times of thermal explosive systems can be understood analytically by using our stochastic path perturbation approach (SPPA). In this paper we present a general approach for tackling that kind of problem and we focus especially on the analytic expression of the first passage time density (FPTD) for the class of universality posed in equation (1). Thus, in principle, all the moments of the passage time can be analysed. We point out that the time-scale characterizing the escape from the instability is the lifetime of the state calculated as the MFPT [3, 4]. The transient relaxation of the system is also studied, i.e. the anomalous fluctuations of the phase-space variable (the moments of the order parameter $\langle X(t)^n \rangle$) are calculated. We have also made a comparison with Monte Carlo simulations which show an excellent agreement with our theoretical predictions.

This paper is organized as follows. In section 2 we review the SPPA, study the marginal case $b = 0$, calculate the FPTD and its moments for $b \neq 0$, and we also compare the theory against simulations. In section 3 we introduce an instanton-like approximation in order to study the anomalous fluctuation of the order parameter $X(t)$. In section 4 we sketch the method used for our simulations. Finally, in section 5 we introduce a discussion and present our future research programme.

2. The stochastic path perturbation approach

2.1. Theory

When the process $\xi(t)$ is a Gaussian white noise, the standard theory of a stochastic process gives the FPTD, associated with equation (1), by solving the corresponding adjoint Fokker–Planck operator [8]. The first and second moments of the FPTD from a small domain around $X(t = 0) < 0$ have been studied by using the Fokker–Planck techniques [9]. An alternative for characterizing the time-scale of the escape process comes from the definition of a certain nonlinear relaxation time [10], which in fact has been proved to be of the order of the MFPT [9].

The problem presented in this paper is the characterization of the time-scale of the escape process by looking at each stochastic realization of equation (1). In this way we are going to define a random escape time, t_e, as the random time when amplitude $X(t)$ diverges. This means that the time $t_e = t_e(a, b, \epsilon, \Omega)$ is going to be a function of a random number, Ω, which will be correctly characterized by a certain probability measure $P(\Omega)$.

Then, in principle, all the moments of t_e can be calculated by taking the mean value over the probability measure $P(\Omega)$. This picture has the advantage over the usual Fokker–Planck technique because it displays the existence of the relevant physical parameter of the system $K = b^3/(a\epsilon^2)$ in a direct way†. Depending on the sign and value of K, the system will have different behaviour; here we will only be interested in the case $K > 0$. Alternatively, *our picture allows the analytic calculation of the* FPTD. *Fluctuations of the paths will be* seen to be non-symmetric. The uniqueness of the escape time t_e will also be shown. The FPTD will be obtained from the transformation of the random variable theorem and the FPTD will be shown to be a non-symmetric broad distribution peaked around $\tau = \sqrt{2/ab}$.

In order to introduce a perturbation theory it is convenient to write the process $X(t)$ as the ratio of two stochastic processes:

$$X(t) = \frac{H(t)}{Y(t)}. \tag{2}$$

Using the nonlinear transformation (2) in equation (1) we obtain an equivalent set of coupled equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) = bY(t) + \sqrt{\epsilon}Y(t)\xi(t) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = -aH(t) \tag{4}$$

where

$$\langle\xi(t)\rangle = 0 \qquad \text{and} \qquad \langle\xi(t)\xi(t')\rangle = \delta(t-t').$$

Here the initial conditions are $H(0) = X(0) = 0$ and $Y(0) = 1$. In the absence of noise ($\epsilon = 0$) from equations (3) and (4) we obtain $\frac{\mathrm{d}^2}{\mathrm{d}t^2}Y(t) = -abY(t)$, which is in agreement with the dynamics of the deterministic system. For small ϵ an approximate solution of the coupled equations (3) and (4) can be considered to be approaching $Y(t)$ in equation (3). At the initial noise-diffusive regime in which $Y(t)$ is close to its initial value, $H(t)$ is essentially a Wiener process plus a drift. Hence, we obtain

$$H(t) \cong bt + \sqrt{\epsilon}W(t) \tag{5}$$

where

$$W(t) = \int_0^t \xi(t')\,\mathrm{d}t' \tag{6}$$

is the Wiener process, and $W(0) = 0$ must be used. In order to find an iterative solution, starting with $Y(0) = 1$, we solve equation (4) with the approximate solution of $H(t)$ given by equation (5):

$$\begin{aligned} Y(t) &\cong 1 - a\int_0^t \left[bt' + \sqrt{\epsilon}W(t')\right]\mathrm{d}t' \\ &\cong 1 - \tfrac{1}{2}abt^2 - a\sqrt{\epsilon}\Omega(t) \end{aligned} \tag{7}$$

where the stochastic process $\Omega(t)$ is defined by

$$\Omega(t) = \int_0^t W(t')\,\mathrm{d}t'. \tag{8}$$

Thus, $\Omega(t)$ is a renormalized Gaussian process (see appendix).

† Note that our universal parameter $K = b^3/(a\epsilon^2)$ is related to the parameter k of [9] by $K = k^3$.

Hence, our first approximation for the stochastic path $X(t)$ is

$$X(t) \cong \frac{bt + \sqrt{\epsilon}W(t)}{1 - (abt^2)/2 - a\sqrt{\epsilon}\Omega(t)}. \tag{9}$$

At this level the complicated mechanism of the escape process can be noticed. The numerator is a Wiener process (of $O(\sqrt{\epsilon})$) at the early initial stage if $b = 0$, otherwise there is competition between the drift and the diffusion. From this perturbation it is easy to observe the non-symmetric fluctuations of the paths. The denominator gives the corrections to the statistics due to the nonlinear contribution in the normal form (1) (i.e. aX^2). Note that the numerator of equation (9) is bounded for $t \neq \infty$, since the Wiener process fulfils $W(t)/t \longrightarrow 0$ for $t \longrightarrow \infty$ with probability one.

Rescaling time as $s = t'/t$ in the integrals of the Wiener process, we obtain from equation (8)

$$\Omega(t) = t^{3/2}\Omega \tag{10}$$

where $\Omega \equiv \Omega(1)$ is a random variable characterized by the probability measure

$$P(\Omega) = \sqrt{\frac{3}{2\pi}}\exp(-3\Omega^2/2). \tag{11}$$

(See appendix.) The escape time, defined by $X(t_e) = \infty$, can then be obtained as the zero of the denominator of the stochastic path given in equation (9):

$$1 = \tfrac{1}{2}abt_e^2 + a\sqrt{\epsilon}\Omega t_e^{3/2}. \tag{12}$$

Up to this order ($\sqrt{\epsilon}$) the SPPA gives the random escape time, t_e, as a mapping with the random number Ω. The random escape time is found by inverting t_e as a function of Ω.

Note that $P(\Omega)$ is a symmetric renormalized Gaussian probability measure. Nevertheless, from equation (12) it is easy to see that there is a symmetry breaking in $t_e(\Omega)$ (i.e. under the transformation $\Omega \longrightarrow -\Omega$) as can be appreciated from a simple graph of the solution of equation (12). This is a consequence of the symmetry breaking $X \longrightarrow -X$ in the potential $U(X)$ of the normal form (1).

2.2. *The marginal case* ($b = 0$)

From equation (12) the trivial case $b = 0$ gives

$$t_e = (a^2\epsilon\Omega^2)^{-1/3} \tag{13}$$

for the escape time. Then, the MFPT (starting from the initial condition $X(0) = 0$) can be obtained from the statistics of $\langle\Omega^{-2/3}\rangle$. Equation (13) shows that for the marginal case the scaled parameter of the system will be $(a^2\epsilon\ )^{-1/3}$. Using the results of the appendix we immediately obtain

$$\langle t_e\rangle_{b=0} = (a^2\epsilon)^{-1/3}\Gamma(1/6)(3/2)^{1/3}/\sqrt{\pi} \tag{14}$$

for the MFPT.

In general, higher momenta will diverge as can be seen from the mean values of $\langle\Omega^{-m}\rangle$ if $m > 1$, indicating the occurrence of an anomalous broad FPTD which, in fact, goes as $P(t_e) \propto t_e^{-2}\exp\left(-\frac{3}{2}(a^2\epsilon t_e^3)^{-1}\right)$. This is a consequence of the flatness at the unstable point $X = 0$ (the marginal case).

If $b \neq 0$, equation (12) cannot be easily inverted as $t_e = t_e(a, b, \epsilon, \Omega)$ (uniqueness of a positive t_e is simple to see from a hand graph of equation (12)). Therefore, in the next section we will work out $P(t_e)$ by looking at the Jacobian of the transformation $|d\Omega/dt_e|$.

2.3. The first passage time density

In order to obtain the probability density of the escape times (i.e. the probability that amplitude $X(t)$ diverges between t_e and $t_e + dt_e$), we begin with the relation between Ω and t_e expressed in equation (12). Knowing $P(\Omega)$ and using the transformation of the random variables theorem, it is possible to calculate the FPTD $P(t_e)$ as

$$P(t_e) = P(\Omega)\left|\frac{d\Omega}{dt_e}\right|. \tag{15}$$

Defining the deterministic escape time as

$$\tau = \sqrt{\frac{2}{ab}} \tag{16}$$

and using the fact that the secular equation for t_e naturally introduces the physical parameter $K = b^3/(a\epsilon^2)$†, we find the following expression for the FPTD as a function of the only two parameters K and τ:

$$P(t_e) \equiv P_K(\tau, t_e) = \sqrt{\frac{3}{2\pi}} \exp\left[-\frac{3}{2}\frac{\tau^3}{t_e^3}\sqrt{\frac{K}{8}}\left(1 - \frac{t_e^2}{\tau^2}\right)^2\right]\left|\frac{d\Omega}{dt_e}\right| \tag{17}$$

with

$$\left|\frac{d\Omega}{dt_e}\right| = \frac{1}{2\tau}\left(\frac{K}{8}\right)^{1/4}\left[\left(\frac{\tau}{t_e}\right)^{1/2} + 3\left(\frac{\tau}{t_e}\right)^{5/2}\right]. \tag{18}$$

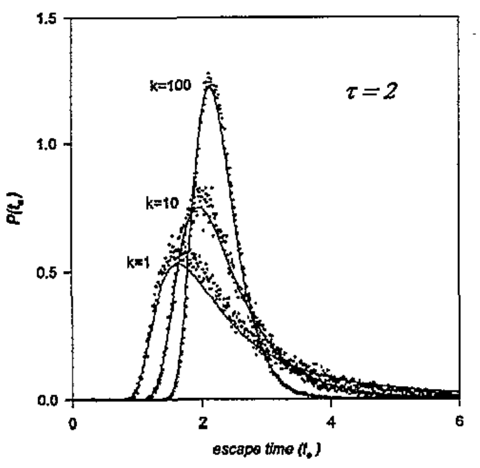

Figure 1. Plot of FPTD $P_K(\tau = 2, t_e)$ as a function of t_e for several values of K (=1, 10, 100). The dots show the Monte Carlo simulations of the SDE (1).

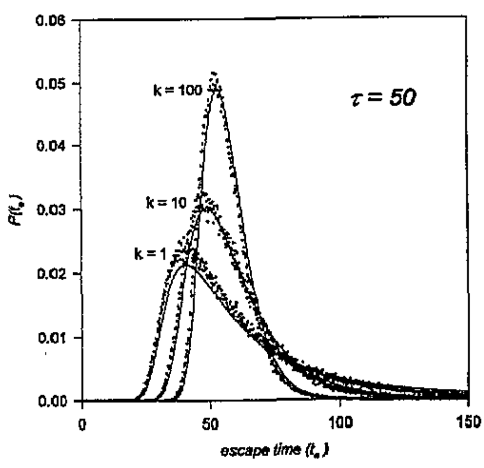

Figure 2. Plot of FPTD $P_K(\tau = 50, t_e)$ as a function of t_e for several values of K (=1, 10, 100). The dots show the Monte Carlo simulations of the SDE (1).

Figures 1 and 2 depict the $P(t_e)$ curves for different values of τ and K. Also, the corresponding Monte Carlo simulations are shown for the same set of parameters.

† Equation (12) for t_e can be transformed into a random algebraic polynomial of degree 4, where Ω is the random variable (Gaussian). The discriminant of the roots leads, naturally, to the definition of K as was pointed out in section 2.1. See also [5].

From the structure of equations (17) and (18) we obtain the scale invariance property

$$P_K(\alpha\tau, \alpha t_e) = \frac{P_K(\tau, t_e)}{\alpha} \tag{19}$$

where α is any arbitrary length scale†. This property can also be observed by the similar appearance of figures 1 and 2, where the relation of equation (19) is satisfied with the scaling $\alpha = 25$.

The approximation described above has a systematic underestimation of the escape time [3]. To solve this problem we introduce a simple modification to our approach. From equations (3) and (4) we find

$$\frac{d^2}{dt^2}Y(t) = -abY(t) - a\sqrt{\epsilon}Y(t)\xi(t). \tag{20}$$

This equation can be identified with the *Kubo oscillator* [8]. Due to the fact that $\xi(t)$ is a Gaussian white noise, it is possible to obtain an exact equation for the mean value of $Y(t)$:

$$\frac{d^2}{dt^2}\langle Y(t)\rangle = -ab\langle Y(t)\rangle. \tag{21}$$

By introducing the initial condition $\langle Y(0)\rangle = 1$, the solution of equation (21) is $\langle Y(t)\rangle = \cos(\sqrt{ab}t)$. The mean value vanishes when $t_m = \pi/\sqrt{4ab}$, independently of the noise parameter ϵ, while our approximation, equation (12), with $\epsilon = 0$ gives a different escape time $\tau = \delta\ t_m$, with $\delta = \sqrt{8}/\pi$. A simple way to improve our result is to force equation (12) to vanish at the correct time t_m for arbitrary ϵ. This is done by introducing the quantity δ by

$$1 = \tfrac{1}{2}ab(\delta t_e)^2 + a\sqrt{\epsilon}\Omega(\delta t_e)^{3/2}. \tag{22}$$

Thus, redefining the physical constants as $a' = \delta^{3/2}a$ and $b' = \delta^{1/2}b$, the formulae for $P_K(\tau, t_e)$, K and τ remain valid. In this way we improve the FPTD given by equation (17). Figures 1 and 2 shows the result of taking this procedure into account.

2.4. *Moments of the FPTD*

The first and second cumulants of the FPTD, $P(t_e)$, are

$$\langle t_e\rangle = \int_0^\infty t_e P(t_e)\, dt_e \tag{23}$$

and

$$\langle(t_e - \langle t_e\rangle)^2\rangle = \int_0^\infty (t_e - \langle t_e\rangle)^2 P(t_e)\, dt_e. \tag{24}$$

From equation (17) these cumulants can be analytically calculated as

$$\langle t_e\rangle = \tau F_1(K) \tag{25}$$

where

$$F_1(K) = \sqrt{\frac{3}{8\pi}}\left(\frac{K}{8}\right)^{1/4}\int_0^\infty \exp\left(-\frac{3}{2}\sqrt{\frac{K}{8}}\frac{(1-\mu^2)^2}{\mu^3}\right)[3\mu^{-2}+1]\sqrt{\mu}\, d\mu. \tag{26}$$

Alternatively,

$$\langle(t_e - \langle t_e\rangle)^2\rangle = \tau^2\left(F_2(K) - F_1^2(K)\right) \tag{27}$$

† Note that K is invariant under the scale transformation $X \to lX$, $t \to st$ of equation (1), therefore equation (19) follows.

with

$$F_2(K) = \sqrt{\frac{3}{8\pi}}\left(\frac{K}{8}\right)^{1/4}\int_0^\infty \exp\left(-\frac{3}{2}\sqrt{\frac{K}{8}}\frac{(1-\mu^2)^2}{\mu^3}\right)[3\mu^{-2}+1]\mu^{3/2}\,\mathrm{d}\mu. \tag{28}$$

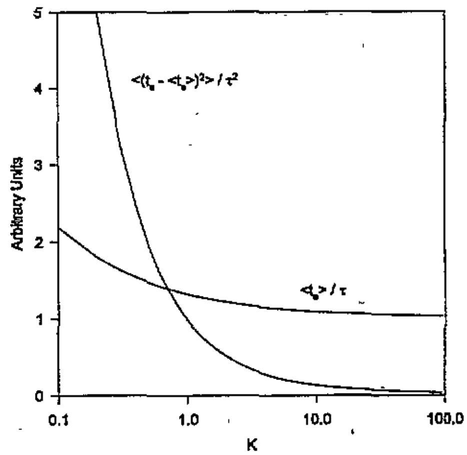

Figure 3. Plot of the dimensionless mean value and variance of $P_K(\tau, t_e)$ as a function of $K = b^3/(a\epsilon^2)$, the universal parameter of the normal form (1).

In figure 3 we show $\langle t_e\rangle/\tau$ and $\langle(t_e - \langle t_e\rangle)^2\rangle/\tau^2$ as functions of the universal parameter K. This figure shows that the MFPT goes to τ as soon as $K \gg 1$; alternatively, the variance (second cumulant) goes to zero in the same limit. Note that this limit is achieved for small noise (for fixed a and b). This means that the FPTD goes to a peaked density centred around τ with a very narrow width. We stress that using the initial condition $X(0) = 0$ our results, given in equations (25) and (27), are beyond the scope of the Colet *et al* [9] analysis . We wish to remark that for the initial condition $X(0) = 0$, our theoretical predictions are in excellent agreement with the Monte Carlo simulation as can be readily seen from figures 1 and (2).

3. Transient fluctuations

In this section we basically follow the work done in [3]. The transient fluctuation in the phase-space variable is defined as the mean quadratic deviation of the $X(t)$ process [11]:

$$\Delta(t) = \langle X^2\rangle - \langle X\rangle^2. \tag{29}$$

In order to calculate the anomalous fluctuation, a saturation term in the normal form equation (1) must be incorporated. Thus, we approximate the transient toward a global attracting solution by introducing the instanton-like approximation

$$X(t) = x_0\Theta(t - t_e) \tag{30}$$

with x_0 the O(1) macroscopic amplitude of the space variable (characterizing the attractor), and $\Theta(t-t_e)$ the Heaviside step function. Taking $x_0 = 1$, the transient anomalous fluctuation is given by

$$\Delta(t) = A(t)(1 - A(t)) \tag{31}$$

where

$$A(t) = \langle \Theta(t - t_e) \rangle = \int_0^\infty \Theta(t - t_e) P_K(\tau, t_e)\, dt_e = \int_0^t P_K(\tau, t_e)\, dt_e. \qquad (32)$$

Thus, we obtain, after using equation (17),

$$A(t) = \frac{1}{2}\mathrm{erfc}\left[\frac{3}{2}\left(\frac{\tau}{t}\right)^{3/2}\left(\frac{K}{8}\right)^{1/4}\left(1 - \frac{t^2}{\tau^2}\right)\right]. \qquad (33)$$

In this instanton-like approximation the maximum of the function $\Delta(t)$ is at $t = \tau$. For fixed a, b the width of $\Delta(t)$ increases with the strength of the noise as can be seen from equations (31) and (33). If we scale t and τ by a length α, the transient anomalous fluctuation remains unchanged, so we can conclude that a scale-invariance transformation is also present in $\Delta(t)$.

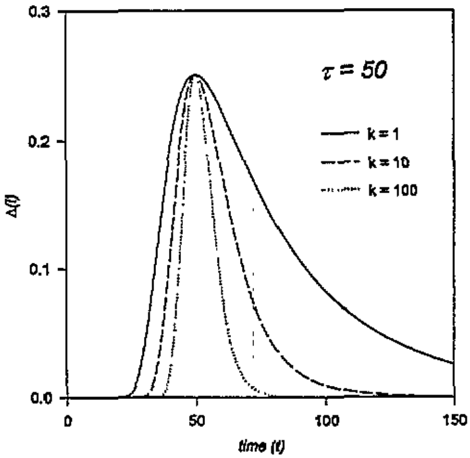

Figure 4. Plot of anomalous transient fluctuations $\Delta(t) = \langle X(t)^2 \rangle - \langle X(t) \rangle^2$ as a function of t for $\tau = 50$ and several values of K (=1, 10, 100). Here, the instanton-like approximation has been used.

Figure 4 depicts the $\Delta(t)$ curve for $\tau = 50$ and different values of K. In the transient regime the initial fluctuations are amplified and give rise to the transient anomalous fluctuations [4, 11] of O(1) as compared with the initial or final fluctuations of $O(\sqrt{\epsilon})$.

4. Monte Carlo simulations

We have accomplished Monte Carlo simulations of the SDE, equation (1), to check the accuracy of our approach. From these simulations we have obtained the histogram of $P_K(\tau, t_e)$ shown in figures 1 and 2. We have used a Heun algorithm [12] that discretizes the SDE (1):

$$x(t_{i+1}) = x(t_i) + \frac{h}{2}\left[a(x^2(t_i) + \hat{x}^2(t_{i+1})) + 2b\right] + \sqrt{\epsilon h} w_i \qquad (34)$$

with the predicator step

$$\hat{x}(t_{i+1}) = x(t_i) + h(ax^2(t_i) + b) + \sqrt{\epsilon h} w_i. \qquad (35)$$

Here, h is the time step $h = t_{i+1} - t_i$, and w_i are independent Gaussian distributed random variables with zero-mean value and variance one. These random numbers are generated using the Box–Muller method [13].

We record the escape time t_e at which $x(t_i) \geqslant x_0$, the escape position for the first time. This procedure is repeated N times to get the histograms. To obtain the results shown in the figures we have used this procedure with $N = 200\,000$ and $h = 0.01$.

5. Discussion

This paper is inspired by a method recently developed and already successfully applied to study relaxation from a subcritical pitchfork bifurcation [3]. In a previous paper [5] it was shown that the stochastic Semenov model leads to the normal form (1). Therefore, its MFPT (i.e. the lifetime of the state) is a relevant quantity to study thermal explosive systems near the critical point [7].

In this paper we have analytically found that the FPTD leaves the unstable state $X(0) = 0$ of the SDE (1). In section 2.1 we introduced the SPPA, and the stochastic paths have been obtained up to $O(\sqrt{\epsilon})$. Figures 1 and 2 show a very good agreement with the Monte Carlo simulation if $K > 1$. For small values of $K = b^3/(a\epsilon^2)$ (the universal parameter of our normal form) the agreement is not so good because for fixed a, b, small values of K means a large noise and, therefore, our paths (equation (9)) start to fail. In section 2.2 the marginal case $b = 0$ was discussed. In that particular case all the fractional moments of the passage times were given in terms of the statistics of $\langle \Omega^{-m} \rangle$. Also, the natural dimensionless parameter of the system was shown to be $(a^2\epsilon)^{-1/3}$. In particular, at the marginal case, the FPTD has a long tail characterized by the power-law asymptotic from $P(t_e) \approx t_e^{-2}$ for $t_e \longrightarrow \infty$. A more interesting situation is when $b \neq 0$; in this case equation (17) gives the desired result. A remarkable result found from the FPTD $P_K(\tau, t_e)$ is the scale invariance property given in equation (19), which becomes a useful tool to analyse experimental results. The moments of the FPTD $P_K(\tau, t_e)$ were analytically calculated (see section 2.4) showing the expected behaviour (figure 3), i.e. for fixed a, b, as soon as the noise decreases ($K \gg 1$) the MFPT goes to the deterministic value τ and the width of the FPTD goes narrow.

The transient anomalous fluctuation characterizes the transition from $O(\sqrt{\epsilon})$ to $O(1)$ in the phase-space variable $X(t)$ [4, 11]. This phenomenon occurs when the normal form has a saturation in its potential (i.e. the relaxation toward an attracting solution). We have studied this anomalous behaviour in the variance $\Delta(t) = \langle X(t)^2 \rangle - \langle X(t) \rangle^2$ by introducing an instanton-like approximation for the stochastic realization of the full system (see equation (30)). Figure 4 shows this phenomenon which also depicts the scale-invariance property.

Among the interesting phenomena to be studied are the thermal explosive times in non-homogeneous physicochemical reactors [7]. In this situation we have to take into account the spatial dependence in the order parameter X (the temperature), which appears in the normal form (1). Our SPPA can also be implemented to tackle this problem. Work along this line is in progress.

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Appendix

From the definition of equations (6) and (8) it is simple to see that $\Omega(t)$ is a zero-mean Gaussian process. Its generating functional can be calculated using the cumulant expansion techniques [8] of the Wiener process:

$$G_\Omega([k(t)]) = \left\langle \exp\left(\int ik(t)\Omega(t)\,\mathrm{d}t \right) \right\rangle_{\text{Wiener}} = \exp\left[-\frac{1}{2} \int\int k(t)k(s) \left(\frac{t^3+s^3}{6} \right) \mathrm{d}t\,\mathrm{d}s \right].$$

Therefore, the probability measure of the random variable $\Omega \equiv \Omega(1)$ is

$$P(\Omega) = \sqrt{\frac{3}{2\pi}} \exp(-\tfrac{3}{2}\Omega^2).$$

Using this probability measure it is easy to obtain all the moments of Ω:

$$\langle \Omega^m \rangle = \frac{\Gamma(m+1/2)}{\sqrt{\pi}} \left(\frac{2}{3} \right)^{m/2}.$$

By symmetry, all the odd moments are zero. Note that the inverse moments $\langle \Omega^{-n} \rangle$ are divergent quantities if $n > 1$.

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